

Family Learning: Non-Parametric Statistical Inference with Parametric Efficiency

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November 29, 2017

Abstract

Hypothesis testing and other statistical inference procedures are most efficient when a reliable low-dimensional parametric family can be specified. We propose a method that learns such a family when one exists but its form is not known *a priori*, by examining samples from related populations and fitting a low-dimensional exponential family that approximates all the samples as well as possible. We propose a computationally efficient spectral method that allows us to carry out hypothesis tests that are valid whether or not the fit is good, and recover asymptotically optimal power if it is. Our method is computationally efficient and can produce substantial power gains in simulation and real-world A/B testing data.

1 Introduction

In most statistical problems, parametric modeling assumptions bring many benefits: they simplify the problem, guide us in deriving efficient inference procedures, and help us to summarize and understand the data parsimoniously. These benefits, however, must be weighed against the risk that if we choose a badly misspecified model, the resulting inferences may be highly misleading.

This article proposes a method to *learn* a good parametric model from the data when we face a repeated task analyzing many statistical experiments with similar underlying characteristics. Our motivating insight is that we can treat similar experiments as arising from a common parametric model, each one corresponding to a different parameter value. If a smooth parametric model exists, our method successfully estimates and exploits it to derive an optimal hypothesis testing procedure for the next experiment, asymptotically achieving the same efficiency as if the model were known *a priori*. In addition, these tests retain their advertised significance level under nonparametric assumptions.

*Much of the work was done while a contractor at Facebook.

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1.1 A/B Testing

The original motivation for this work is *A/B testing*: randomized experiments conducted at internet companies to evaluate user behavior under various treatment conditions, typically minor modifications of the website’s behavior. Most large internet companies carry out many A/B tests each day as a way to test new features or check that routine updates have not broken the site. Companies also want to understand how people are engaging with their products so that they can make informed and effective decisions about how to improve upon them. They do so by examining metrics about user engagement such as the time spent on the site or number of photos liked. The repeated nature of this task presents an opportunity for the company to learn from past experiments in order to improve their inferences for the next one.

Despite a common misconception that statistical power is always extremely high when analyzing internet-scale data sets, statistical efficiency is in fact at a high premium in A/B testing for several reasons. First, many of the user metrics that firms use for evaluation have characteristically heavy right tails, making nonparametric estimators like the sample mean highly inefficient. Second, the changes observed can be small, but many small incremental changes can have a cumulatively large effect. For example, detecting an additional 200 changes that each raise user engagement by 0.2% would increase engagement by 50% overall. Third, it is important that the methods are powerful enough to withstand slicing the data into subgroups for more detailed analyses, or correcting for multiple comparisons across many experiments. Finally, companies prefer to involve as few users as possible in each experiment.

1.2 Notation and Problem Setting

We consider an A/B testing problem with m independent treatment groups, obtained by combining many experiments with several treatment groups each, all drawn from a common user population and measuring the same outcome of interest. The i th treatment group is $X_{i,1}, \dots, X_{i,n_i} \stackrel{\text{i.i.d.}}{\sim} P_i$, with $n_+ = \sum_i n_i$ denoting the total number of observations.

We introduce a further working assumption that the distributions P_i arise from a common smooth parametric family of densities $\mathcal{P} = \{p_\theta : \theta \in \Omega \subseteq \mathbb{R}^k\}$ with respect to a common dominating measure ν on the outcome space \mathcal{X} ; typically $\mathcal{X} \subseteq \mathbb{R}$ and ν is the usual Lebesgue measure. We emphasize that the functional form of p_θ is unknown. Let $\mathbb{E}_\theta[\cdot]$, $\text{Var}_\theta[\cdot]$, $\text{corr}_\theta(\cdot, \cdot)$ etc. denote moments computed with respect to p_θ , and let θ_i denote the parameter for group i ($dP_i = p_{\theta_i} d\nu$).

We consider a local asymptotic regime in which the sample sizes grow while the effect sizes shrink, a good match for Facebook data. That is, we assume Ω contains an open neighborhood of 0 and $\theta_i \rightarrow 0$. Locally, it will be sufficient to estimate a local approximation $p_\theta \approx p_0(1 + \theta'T)$, where $T(x)$ is the efficient score function at $\theta = 0$. We assume the regularity condition that \mathcal{P} is *differentiable in quadratic mean* (DQM) at $\theta = 0$ and that the Fisher information $\mathbb{E}_0 T T'$ is non-singular. Under standard regularity conditions the score is more familiarly the gradient of the log-likelihood $T(x) = \nabla_\theta \log p_\theta(x)|_{\theta=0}$. DQM is a weaker condition not requiring differentiability of $\log p_\theta$ at 0. While $\mathbb{E}_0 T(X) = 0$ by necessity, $T(x)$ is only identifiable up to invertible linear transformations. To resolve the ambiguity, we assume without loss of generality that $\text{Var}_0 T(X) = I_k$ (even then, $T(X)$ is only identifiable up to orthogonal transformations).

Given estimates \hat{T} and \hat{p}_0 , we approximate \mathcal{P} using the exponential family model $\hat{\mathcal{P}} = \{e^{\theta'\hat{T}(x) - \psi(\theta)} \hat{p}_0(x) : \int_{\mathcal{X}} e^{\theta'\hat{T}} \hat{p}_0 < \infty\}$. As we will see, applying the estimated model $\hat{\mathcal{P}}$ to future experiments gives nearly the same statistical efficiency as knowing the actual parametric model. Because we are estimating an exponential family approximation, we will refer to $\hat{T}(x)$ as the *sufficient statistic* and $\hat{p}_0(x)$ as the *base measure*, reflecting the roles they play in $\hat{\mathcal{P}}$.

As a concrete example, consider the Laplace location family, with $p_\theta(x) = \frac{1}{2}e^{-|x-\theta|}$. This is not an exponential family, but $p_\theta(x) \approx p_0(x)e^{\theta \text{sgn}(x)}$ for small values of θ . Therefore \mathcal{P} is locally

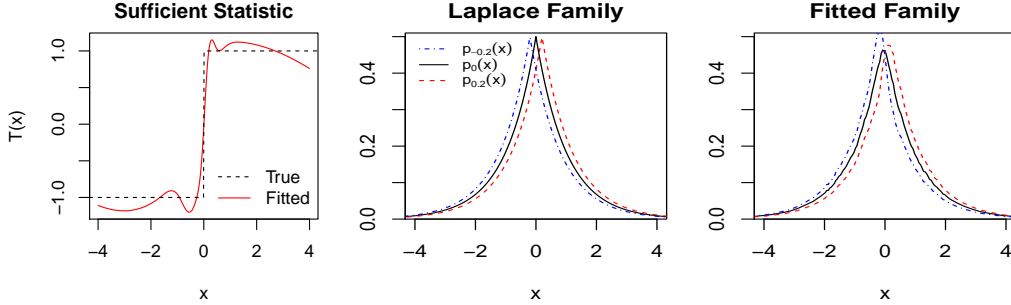


Figure 1: Left: Fitted sufficient statistic for Laplace simulation. Middle: Laplace density for $\theta = -0.2, 0$, and 0.2 . Right: Fitted densities for the same parameter values.

well-approximated by an exponential family with sufficient statistic $T(x) = \text{sgn}(x)$. Figure 1 shows the results of a simulation (revisited in Section 4.1) that illustrates our method. By estimating $T(x)$ and $p_0(x)$, we obtain a one-parameter exponential family that we can subsequently use to analyze experiments as though we had prior knowledge of the true parametric form.

Section 2 details our method for estimation of \mathcal{P} and proves consistency. Section 3 discusses how to perform inference once we have our estimate $\hat{\mathcal{P}}$, and Sections 4–5 illustrate and evaluate our method using simulated and real data from A/B tests performed at Facebook. Section 6 concludes.

1.3 DQM Families and Efficient Hypothesis Testing

While there are many possible uses for the estimated family $\hat{\mathcal{P}}$, we focus on hypothesis testing in a new experiment, i.e. testing $H_0 : P_{m+1} = P_{m+2}$. In particular, for any candidate transformation $U(x)$ with variance $\sigma_U^2 < \infty$ we could use the test statistic

$$\Delta_U = \sqrt{\frac{n_{m+1}n_{m+2}}{(n_{m+1} + n_{m+2})\hat{\sigma}_U^2}} \left[\frac{1}{n_{m+1}} \sum_{j=1}^{n_{m+1}} U(X_{m+1,j}) - \frac{1}{n_{m+2}} \sum_{j=1}^{n_{m+2}} U(X_{m+2,j}) \right], \quad (1)$$

where $\hat{\sigma}_U^2$ is the pooled sample variance. We can find the critical value either by permutation (giving an exact test in finite samples) or by using the more computationally convenient normal approximation $\Delta_U \xrightarrow{H_0} \mathcal{N}(0, 1)$, which gives the right significance level in large samples. These procedures are fully non-parametric in the sense that they yield an asymptotically correct critical value without making any parametric assumptions on the underlying distribution.

We now briefly review the asymptotic theory of hypothesis testing in the case $k = 1$; for a full treatment see Van der Vaart (2000). The differentiable in quadratic mean (DQM) condition is that the square root of the density can be approximated as $\sqrt{p_\theta(x)} = \sqrt{p_0(x)}(1 + \theta'T(x)/2) + r_\theta(x)$, where the remainder term satisfies $\|r_\theta\|_{L_2(\nu)}^2 = \int_{\mathcal{X}} r_\theta(x)^2 d\nu(x) = o(\|\theta\|^2)$. The DQM assumption is a weak and easily satisfied condition and includes ill-behaved distributions such as power law distributions. If the experiment's samples are from a DQM family \mathcal{P} with score $T(X)$ at 0, then the (one-sided) test based on Δ_T is asymptotically equivalent to the uniformly most powerful likelihood ratio test. We analyze power in the local asymptotic regime, where the parameters for the two samples becomes closer as the sample size grows, $\theta_{m+1} = t_{m+1}/\sqrt{n_{m+1}}$ and $\theta_{m+2} = t_{m+2}/\sqrt{n_{m+2}}$; this scaling ensures that the test is neither powerless nor perfectly powerful in the limit as $n_{m+1}, n_{m+2} \rightarrow \infty$. Under this scaling, Le Cam's Third Lemma implies that for almost any candidate U , we have

$$\Delta_U \rightarrow \mathcal{N}\left(\frac{t_{m+1} - t_{m+2}}{2} \text{corr}_0(U, T), 1\right). \quad (2)$$

We make this relationship more explicit in section 3. Note that if $\text{corr}_0(U, T) = 1/2$, we would need to quadruple the size of our sample to achieve the same power using the test statistic Δ_U that we could have achieved by using Δ_T instead. In that sense, the test based on Δ_U uses the data only 25% as efficiently as the test based on Δ_T ; we say its *Pitman relative efficiency* is 0.25 relative to the better test, meaning that testing with Δ_U is asymptotically equivalent to first discarding 75% of the data and then testing with Δ_T . Indeed, Le Cam’s Third Lemma implies that the Pitman relative efficiency of a test based on *any* asymptotically normal test statistic can be measured by its squared correlation to Δ_T under p_0 , and the score T defines the asymptotically efficient test statistic.

1.4 Related work

Hastie and Little (1987) propose a method to decompose contingency tables into low-dimensional multinomial families. Their goal closely resembles ours in spirit in the case of discrete data with a saturated basis S . To our knowledge, our work is the first to propose using a meta-analysis of low powered experiments in order to obtain procedures that are asymptotically efficient, and in particular, to do so non-parametrically. There is a rich body of work on non-parametric two-sample tests in a single experiment. Some tests, such as the t, Mann-Whitney, or Wilcoxon tests, are not guaranteed to have power even in the limit of infinite data, while others such as the Kolmogorov-Smirnov and Anderson-Darling tests, the Kernel Two-Sample Test of Gretton et al. (2012), and the B-test of Zaremba et al. (2013) do. For finite samples choosing the appropriate test can be difficult as it is not clear which test is most powerful for a given problem.

Unlike most non-parametric hypothesis tests, our method also allows for inference about functionals $\phi(F)$ of a distribution. These include the mean, median, or other interpretable quantities of interest. Semi-parametric methods such as those by van der Laan and Rubin (2006) and Klaassen (1987) can be used when there is target functional or parameter of interest without resorting to strong parametric assumptions. Surprisingly, they can give asymptotically efficient estimators. However, the efficiency depends on eventually having a high signal-to-noise ratio, an unrealistic assumption when the goal is to improve inference for experiments with low power. We instead combine many experiments with low signal to improve efficiency.

Fithian and Wager (2015) and Taddy et al. (2016) propose methods to improve mean estimates in A/B testing by using a large background dataset to better estimate the tails for heavy tailed distributions.

2 Estimating the Family \mathcal{P}

Our method for estimating \mathcal{P} proceeds in two steps. First, we employ a spectral method to estimate $T(x)$, which requires no knowledge or estimate of $p_0(x)$. Second, we use the fitted $\hat{T}(x)$ to estimate $p_0(x)$ using a Poisson generalized linear model (GLM). For simplicity we focus our analysis on the one-parameter case; $k > 1$ is similar but with more notational overhead.

2.1 Estimating the Sufficient Statistic $T(x)$

Let $(S_i(x))_{i=1}^{\infty}$ denote a dense basis of $L^2(p_0)$, specified in advance and with each S_i bounded. For a fixed dimension d , we approximate T as a linear combination of the first d basis vectors $S(x) = (S_1(x), \dots, S_d(x))$. Truncating the basis to d elements incurs an approximation error shrinking to 0 as $d \rightarrow \infty$.

Let $W = \text{Var}_0 S(X) \in \mathbb{R}^{d \times d}$. Define the group mean $\bar{S}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} S(X_{i,j})$, the grand mean

$\bar{S} = \frac{1}{n_+} \sum_{i=1}^m n_i \bar{S}_i$, and the empirical between- and within-treatment-group covariance matrices:

$$\hat{A} = \frac{1}{m-1} \sum_{i=1}^m n_i (\bar{S}_i - \bar{S})(\bar{S}_i - \bar{S})', \quad \text{and} \quad (3)$$

$$\hat{W} = \frac{1}{n_+ - m} \sum_{i=1}^m \sum_{j=1}^{n_i} (S(X_{i,j}) - \bar{S}_i)(S(X_{i,j}) - \bar{S}_i)'. \quad (4)$$

We estimate $\hat{T}(x) = \hat{\beta}' S(x)$, where $\hat{\beta} \in \mathbb{R}^d$ maximizes the Rayleigh quotient $\beta' \hat{A} \beta / \beta' \hat{W} \beta$.

We now state our main theorem: under appropriate assumptions, \hat{T} converges to T in $L^2(p_0)$, up to translation and scaling. Consider a sequence of problems satisfying the following:

Assumption 1. $m, n_+ - m \rightarrow \infty$ (the total degrees of freedom for \hat{W} and \hat{A} are growing)

Assumption 2. $\frac{1}{m} \sum_{i=1}^m n_i (\theta_i - \bar{\theta})^2 = \Omega(1)$ (the average signal strength does not shrink to 0)

Assumption 3. $\max_i |\theta_i| \rightarrow 0$ (effect sizes are small)

Assumptions 4–6 are fairly mild conditions. To grasp them more easily, consider a balanced random effects model where $n_i = n$, and $\theta_i = t_i / \sqrt{n}$ where $t_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\alpha, \sigma^2)$ with α, σ^2 fixed. Then, Assumptions 4–6 hold almost surely provided that $m, n \rightarrow \infty$ with $\log m = o(n)$.

Theorem 1. *Under Assumptions 4–6, if we take $d \rightarrow \infty$ slowly enough, then $\hat{T}(x)$ is consistent for $T(x)$ in $L^2(p_0)$ up to sign; that is, $\text{corr}_0^2(\hat{T}(X), T(X) | \hat{T}) \xrightarrow{P} 1$.*

This proof and all others are deferred to the technical supplement, but the following derivation motivates $\hat{\beta}$ heuristically: Let $\mu_0 = \mathbb{E}_0 S(X)$, $c = \text{Cov}_0(S(X), T(X))$, $D = \mathbb{E}_0[TS S']$, and write

$$\mu_i = \mathbb{E}_{\theta_i} S(X) \approx \int_{\mathcal{X}} S(x) p_0(x) (1 + \theta_i' T(x)) d\nu(x) = \mu_0 + \theta_i c, \quad (5)$$

$$W_i = \text{Cov}_{\theta_i} S(X) \approx W + \theta_i (D + \mu_0 c' + c \mu_0') + \theta_i^2 c c' = W + O(\theta_i) \quad (6)$$

Writing $T_\beta(x) = \beta' S(x)$, we would ideally choose β to maximize correlation with the score:

$$\rho_\beta^2 = \frac{\text{Cov}_0(T_\beta(X), T(X))^2}{\text{Var}_0(T_\beta(X))} = \frac{\beta' c c' \beta}{\beta' W \beta}.$$

Since c and W are unknown, we estimate $\hat{W} \approx W$ and maximize a proxy for ρ_β^2 . If $\bar{\theta} = n_+^{-1} \sum_i n_i \theta_i$ then the expectation of the between-groups covariance matrix \hat{A} is

$$A = \mathbb{E} \hat{A} \approx \frac{1}{m-1} \sum_{i=1}^m n_i (\theta_i - \bar{\theta})^2 c c' + \left(1 - \frac{n_i}{n_+}\right) W_i = \frac{\tau^2}{m-1} c c' + W + O(\max_j |\theta_j|), \quad (7)$$

where $\tau^2 = \sum_{i=1}^m n_i (\theta_i - \bar{\theta})^2$ is the overall signal strength, leading to the objective function:

$$\hat{f}(\beta) = \frac{\beta' \hat{A} \beta}{\beta' \hat{W} \beta} \approx \frac{\beta' A \beta}{\beta' W \beta} = \frac{\tau^2 \rho_\beta^2}{m-1} + 1 = f(\beta) \quad (8)$$

$\hat{f}(\beta)$ is maximized by $\hat{\beta} = \hat{W}^{-1/2} \tilde{\beta}$, where $\tilde{\beta}$ is the leading eigenvector of $\hat{W}^{-1/2} \hat{A} \hat{W}^{-1/2}$.

Note that Theorem 1 does not tell us how fast to take $d \rightarrow \infty$, much less how to choose d in finite samples. A more refined analysis would reveal that the choice depends greatly on quantities like the signal strength τ^2 , the correlation under p_0 of the basis elements, and the smoothness of \mathcal{P} ; since these are all unknown, in practice we recommend choosing d via cross-validation: since

Type I error on held-out data is always α , we can choose d to maximize the number of rejections. Cross-validation can also help choose between competing bases. In our motivating problems we have found natural spline bases with $d = 10$ to 15 sufficient to achieve very high correlations with the true $T(x)$.

The assumption that S is bounded can be weakened to subgaussianity and mild moment assumptions, but these assumptions are unverifiable without knowing \mathcal{P} (unless S is bounded). Still, if practical domain knowledge motivates using a given unbounded basis in a given setting, there is nothing in the theory to suggest our method will fail. Once again, cross-validation is the best guide.

Our method is highly computationally efficient. Most of the work is computing the summary statistics $\bar{S}_1, \dots, \bar{S}_m \in \mathbb{R}^d$ and $\widehat{W} \in \mathbb{R}^{d \times d}$, which can be done in a single parallelizable scan over the entire data set, in time $\mathcal{O}(n_+ d^2)$. The remaining computation to obtain \widehat{T} requires $\mathcal{O}(m d^2 + d^3)$ time.

2.2 Estimating the Base Measure $p_0(x)$

To produce the plots in Figure 1, we use a method following Efron et al. (1996): we first bin the data, computing $N_{i,b} = \#\{j : X_{i,j} \in \text{bin } b\}$, and choosing for each bin a representative value $x_b = \mathbb{E}_0[X \mid X \in \text{bin } b]$. Using $\mathbb{E}N_{i,b} \approx n_i e^{\theta T(x_b)} p_0(x_b)$, we then estimate a Poisson GLM with $\log \mathbb{E}N_{i,b} = \alpha_i + \zeta_b + \theta_i \widehat{T}(x_b)$ and set $\hat{p}_0(x) \propto \sum_b e^{\hat{\zeta}_b} \delta_{x_b}(x)$.

3 Inference with an Estimated Family

After obtaining $\widehat{\mathcal{P}}$, we want to use it for future experiments. The main result in this section is that, if $\text{corr}_0(\widehat{T}(X), T(X) | \widehat{T})^2 \rightarrow 1$, then hypothesis tests in future experiments are asymptotically efficient; in other words, we can do as well as if we knew the true family of distributions \mathcal{P} ahead of time. Theorem 2 considers two-sample testing in a held-out experiment as discussed in Section 1.3.

Theorem 2. *Let $\mathcal{P} = \{p_\theta : \theta \in \Omega \subseteq \mathbb{R}\}$ be DQM at $\theta = 0 \in \Omega^\circ$, with score function $T(x)$ at $\theta = 0$, and consider a sequence of estimators \widehat{T}_n with $\text{corr}_0(\widehat{T}_n(X), T(X) | \widehat{T}_n) \xrightarrow{p} 1$ as $n \rightarrow \infty$, independently of the two samples. If $n_{m+1}, n_{m+2} \rightarrow \infty$ with $\theta_{m+1} = t_{m+1}/\sqrt{n_{m+1}}$ and $\theta_{m+2} = t_{m+2}/\sqrt{n_{m+2}}$, then the test rejecting for large $\Delta_{\widehat{T}_n}$ is asymptotically efficient for testing the one-sided hypothesis $H_0 : \theta_{m+1} \leq \theta_{m+2}$ versus $H_1 : \theta_{m+1} > \theta_{m+2}$.*

Thus, our procedure produces an asymptotically efficient test as long as the appropriate sign can be correctly identified. This is typically simple as one is interested in the mean or some monotone increasing transformation of the metric. One chooses the sign which makes $\text{corr}(\widehat{T}(X), X) > 0$.

The notion of asymptotic efficiency for multidimensional families is more complex as there is no uniformly most powerful test. However, one may still obtain asymptotically efficient tests when examining only one direction of the parameter space. In particular, for testing a difference of means, if one tests for a difference in $\nabla_0 \mu(\theta)(\theta_1 - \theta_0)$ where $\mu(\theta)$ maps the parameters to the mean, the resulting test is asymptotically efficient for testing a one-sided hypothesis for the difference in means.

3.1 Other Inferences

The uses to which we can put our fitted family extend far beyond hypothesis testing. Generally speaking, we can use $\widehat{\mathcal{P}}$ in any of the myriad ways that we can use a parametric model that is specified in advance. The benefit of $\widehat{\mathcal{P}}$ over a prespecified model, however, is that we can proceed

with the confidence that our parametric “assumptions” are actually validated by the data rather than having been chosen by fiat for convenience’s sake.

For example, suppose that we are given a sample $X_{0,1}, \dots, X_{0,n_0}$ from some new population $p_{\theta_0}(x)$, and we wish to estimate some new parameter $\mu(\theta_0)$, say $\mu = \mathbb{E}_{\theta_0} X$. Given a confidence set C_{θ_0} , we can derive a corresponding confidence set for μ via $C_\mu = \{\widehat{\mathbb{E}}_\theta X : \theta \in C_{\theta_0}\}$. In particular, if μ and θ_0 are univariate and C_{θ_0} is an interval, and $\frac{d\mu}{d\theta}(0) \neq 0$, then for small values of θ we can simply transform the endpoints of C_{θ_0} to obtain the endpoints of C_μ . We apply this method in Section 4 to obtain 95% intervals for $\mu = \mathbb{E}_\theta X$, and see that the intervals enjoy their advertised coverage.

4 Simulations

4.1 Laplace Location Family

We revisit the Laplace location family discussed in Section 1.2. We use a natural spline basis with 11 degrees of freedom. Note that the smooth spline basis is intentionally somewhat mismatched to the score $T(x) = \text{sgn}(x)$ that we are trying to learn. This reflects the fact that we cannot rely on choosing a “good” basis in advance for real data. We generate $m = 5000$ groups with $n_i = 100$ data points each, with $\theta_i \sim \mathcal{N}(0, 0.1^2)$.

Despite the mismatched basis, our method learns the score up to correlation $\rho^2 = 93.8\%$. The 95% confidence intervals for θ_i achieve their advertised coverage, with 248 of 5000 failing to cover. The average interval width is 0.4, which is 30% shorter on average than one-sample t -intervals. Although n_i is much too small to reliably distinguish $\theta = 0$ from $\theta = 0.1$ — and would be even if we knew the true parametric form in advance — by combining many groups together we can still learn much richer information, piecing together a very accurate local approximation to the family \mathcal{P} .

4.2 Log-Gamma Family

Next, we evaluate our method on the log-gamma family. That is, $\log(X_{i,j})$ is gamma-distributed with scale parameter 0.4 and shape parameter θ_i . Thus, population i has exponential family density

$$p_{\theta_i}(x) = e^{\theta_i \log(\log(x))} \cdot \frac{x^{-3.5} \log(x)^{-1}}{0.4^{\theta_i} \Gamma(\theta_i)}, \quad \text{for } x > 1.$$

We have chosen the log-gamma family because of its heavy right tail, making it qualitatively similar to some real world metrics. We simulate $m = 100$ groups with $n_i = 5000$ observations each, with parameters θ_i drawn from $\mathcal{N}(3, 0.5^2)$, and use a natural spline basis with 11 degrees of freedom.

Figure 2 shows results from the simulation. In the left panel we plot the fitted sufficient statistic against the (centered and scaled) true score, $1.6 \log(\log(x)) + 0.014$. Intuitively, $T(x)$ is forced to be highly concave because of the log-gamma distribution’s heavy right tail — for example, $T(x) = x$ is completely out of the question because $p_0(x)e^{\varepsilon x}$ is not even normalizable for any $\varepsilon > 0$. Thus, the sample mean is a very poor summary of the data with relative efficiency only 7.7%; in other words, tests based on the sample mean effectively discard over 90% of the data. By contrast, our method obtains an accurate estimate that is 98.6% efficient.

The right two panels of Figure 2 show 95% confidence intervals for the expectation $\mu_i = \mathbb{E}_{\theta_i}[X] = (1 - 0.4)^{-\theta_i}$ of each treatment group computed with our method, alongside t -intervals for the same parameters. The intervals using the fitted family are much shorter despite retaining the correct advertised coverage; this is because they are based on a much more efficient summary of the data.

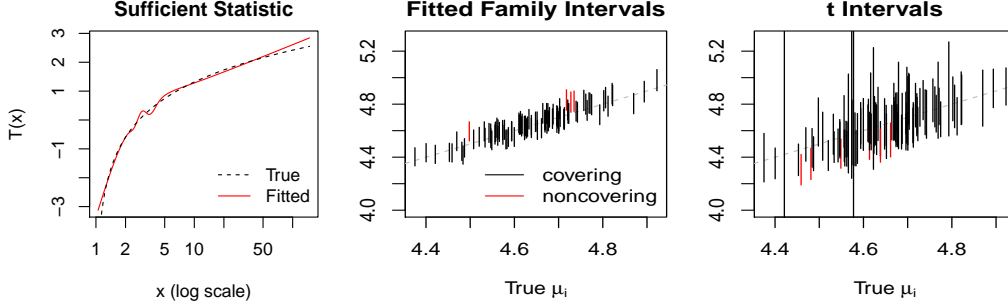


Figure 2: Log-gamma results. Left panel: The fitted sufficient statistic is highly correlated with the true score ($\rho^2 = 98.6\%$). Right two panels: 95% confidence intervals for each μ_i using the fitted family, versus t -intervals for same. The horizontal axis shows the true target μ_i for each interval. Both methods achieve the advertised coverage, but the t -intervals are 2.7 times as long on average.

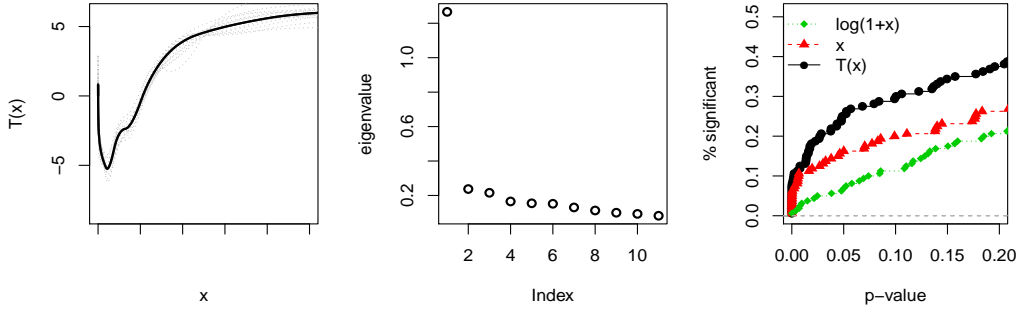


Figure 3: Left: Estimated sufficient statistics for time spent on the Facebook website. The black line shows the estimate using all the data. The gray lines show fits from bootstrapping. Center: Scree plot showing only one obvious sufficient statistic. Right: Cumulative distribution of p -values for tests using original values (red), log-transformed values (green), estimated sufficient statistic (black).

5 Results on Facebook Data

To give an example of our method in action, we demonstrate it on Facebook A/B testing data, which provided the motivation for devising our method. Some changes to Facebook result in small, incremental improvements to metrics of interest. These small differences are difficult to detect and necessitate either larger experiments or more effective measurements. The empirical results bear out that experimental effects behave locally like a low dimensional parameterized family. Furthermore, we are able to obtain a significantly more powerful test. Table 1 shows the estimated relative efficiencies of various statistics compared to using the estimated sufficient statistic.

The data consists of de-identified, aggregated distributions of rescaled outcomes over large, random subsets of users to ensure privacy. In the examples here, the data consists of 195 subpopulations with average sample size exceeding 2×10^6 . The x -axes are rescaled by some constant and the far tails of the distribution are truncated beyond the 99th percentile in the plots.

We show two metrics that can be of interest: number of photo likes and time spent on the Facebook website. These can be used as diagnostic measures for understanding user experience,

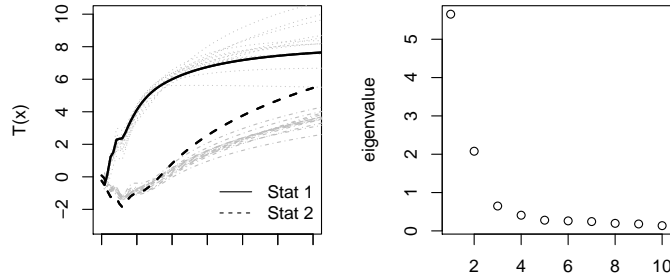


Figure 4: Left: Estimated sufficient statistics for number of photo likes. The black lines shows the estimate using all the data. The grey lines show fits when bootstrapping the data. Center: Scree plot showing two possible sufficient statistics.

Scenario	Test Statistic		
	X	$\log(1 + X)$	True $T(X)$
Log-gamma	0.08	0.85	1.01
Laplace location	0.53	0.71	1.07
Facebook time spent	0.32	0.04	—

Table 1: Asymptotic efficiency of tests relative to estimated sufficient statistic.

for instance to detect bugs in the code or, for time spent, understanding if the page can be improved to provide users’ relevant information more quickly. Both are measured at the user level over some period of time. For the photo likes metric, figure 4 shows that the estimated sufficient statistic has a shape similar to a log transformation. This matches what one would expect to be a good transformation since the distribution of photo likes has a long tail. For the time spent metric, it also down-weights large values and effectively winsorizes them. Unlike the photo likes metric, the estimated sufficient statistic is clearly not monotonic, and we can only speculate as to the causes of the dip. However, the sufficient statistic leads to substantially more powerful tests as shown in figure 3. The proportion of significant tests at the 5% level jumps from 15% to 24%. This difference has a p-value of 0.002 using McNemar’s test. Interestingly, the time spent sufficient statistic also demonstrates that the log transform is not always a good transform even if a distribution has fairly “heavy” tails. In this case the log transform performs worse than the untransformed data. It has thick tails as the standard deviation is high relative to the mean; the coefficient of variation is over 400%.

6 Discussion and Conclusion

We propose a method that non-parametrically learns a low-dimensional parametric family. When applied to A/B testing, this allows combining many low powered experiments to find asymptotically efficient hypothesis tests and estimators. The efficacy of the method is demonstrated on real world A/B test data where it triples the effective sample size relative to a difference in means z-test.

One natural question for further study is how to choose the number of sufficient statistics to estimate. As shown in the scree plots in figures 3 and 4, a good choice is often obvious. More systematically, one can split the data into test and training sets and evaluate a goodness of fit criterion such as a likelihood ratio.

One assumption that can be restrictive is that all treatment groups are drawn from a com-

mon user population. This is necessary for the spectral method that we have proposed in order to obtain an appropriate scaling matrix W . However, this assumption is not necessary if an exponential family model is learned directly. In this case, one fits the collection of models $\mathcal{M}_i = p_{i,0}(x)e^{\theta_{i,j}^T T(x) - \psi_i(x)}$ where i denotes the subpopulation and j the treatment group within that subpopulation. This can be fit by fitting Poisson generalized linear models that alternately estimate the parameters θ and the sufficient statistics T .

It would also be interesting to extend this work to handle multivariate responses. This may be done by extending the basis S from a univariate to a multivariate basis. However, in order to well approximate any sufficient statistic, the number of needed basis elements may grow exponentially in the dimension; structural assumptions such as additivity or sparsity may be essential in this setting.

Acknowledgement: We thank Facebook for allowing us to report results on their dataset. The authors are grateful for stimulating conversations with Jackson Gorham, Stefan Wager, Joe Romano, and Trevor Hastie. William Fithian was supported in part by the Gerald J. Lieberman Fellowship.

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A Preliminaries

We begin by stating a lemma relating quadratic mean differentiability to differentiability of moments, which we will need in the proof.

Lemma 3. Assume that $\mathcal{P} = \{p_\theta : \theta \in \Omega \subseteq \mathbb{R}\}$ is QMD in a neighborhood of θ , and $T \in L^2(p_0)$ is the score function at $\theta = 0$. Let \mathcal{F} be a set of real-valued functions. If there exists $B > 0$ such that for all $f \in \mathcal{F}$, $\mathbb{E}_\theta f^2 < B$ for all θ in a neighborhood of θ , and $\mathbb{E}_0 |f^2 T|, \mathbb{E}_0 |f T^2|, \mathbb{E}_0 f^2 T^2 < B$, then as $\theta \rightarrow 0$ we have

$$\sup_{f \in \mathcal{F}} |\mathbb{E}_\theta - (\mathbb{E}_0 f + \theta \mathbb{E}_0 f T)| = o(|\theta|). \quad (9)$$

Proof. Let $\xi_\theta = \sqrt{p_\theta}, \tilde{\xi}_\theta = \sqrt{p_0}(1 + \theta T/2)$. QMD implies that $\|r_\theta\| = \|\xi_\theta - \tilde{\xi}_\theta\| = o(|\theta|)$ in norm, where the norm is defined with respect to the $L^2(\nu)$ inner product $\langle u, v \rangle = \int_{\mathcal{X}} u(x)v(x) dx$. Then

$$\mathbb{E}_\theta f - \mathbb{E}_0 f = \langle f(\tilde{\xi}_\theta + r_\theta), \tilde{\xi}_\theta + r_\theta \rangle - \langle f\xi_0, \xi_0 \rangle \quad (10)$$

$$= \langle f\tilde{\xi}_\theta, \tilde{\xi}_\theta \rangle - \langle f\xi_0, \xi_0 \rangle + \langle r_\theta, f(r_\theta + 2\tilde{\xi}_\theta) \rangle \quad (11)$$

$$= \langle \theta f T \xi_0, \xi_0 \rangle + \frac{1}{4} \langle f \theta^2 T \xi_0, T \xi_0 \rangle + \langle r_\theta, f(\xi_\theta + \tilde{\xi}_\theta) \rangle \quad (12)$$

$$= \theta \mathbb{E}_0 f T + \frac{\theta^2}{4} \mathbb{E}_0 f T^2 + \langle r_\theta, f(\xi_\theta + \tilde{\xi}_\theta) \rangle \quad (13)$$

$$|\mathbb{E}_\theta f - (\mathbb{E}_0 f + \theta \mathbb{E}_0 f T)| \leq \frac{\theta^2}{4} \mathbb{E}_0 f T^2 + \|r_\theta\| |\mathbb{E}_\theta f^2 + \mathbb{E}_0 [f^2(1 + \theta T/2)]| \quad (14)$$

$$\leq \frac{\theta^2 B}{4} + (2 + \theta/2) B \|r_\theta\|. \quad (15)$$

Because the final bound does not depend on f , the proof is complete. \square

As an immediate corollary, suppose that $\sup_{x \in \mathcal{X}} |S_i(x)| \leq B$ for all $i = 1, \dots, d$. Then as $\theta \rightarrow 0$ we have for $c_i = \mathbb{E}_0 S_i T$ and $D_{ij} = \mathbb{E}_0 S_i S_j T$:

$$\sup_i |\mathbb{E}_\theta S_i - (\mathbb{E}_0 S_i + \theta c_i)| \rightarrow 0 \quad \text{and} \quad \sup_{i,j} |\mathbb{E}_\theta S_i S_j - (\mathbb{E}_0 S_i S_j + \theta D_{ij})| \rightarrow 0.$$

B Proof of Theorem 1

We restate Assumptions 4–6 and Theorem 1 here for convenience.

Assumption 4. $m, n_+ - m \rightarrow \infty$ (the total degrees of freedom for \widehat{W} and \widehat{A} are growing)

Assumption 5. $\frac{1}{m} \sum_{i=1}^m n_i (\theta_i - \bar{\theta})^2 = \Omega(1)$ (the average signal strength does not shrink to 0)

Assumption 6. $\max_i |\theta_i| \rightarrow 0$ (effect sizes are small)

Theorem 1. Under Assumptions 4–6, if we take $d \rightarrow \infty$ slowly enough, then $\widehat{T}(x)$ is consistent for $T(x)$ in $L^2(p_0)$ up to sign; that is, $\text{corr}_0^2(\widehat{T}(X), T(X)|\widehat{T}) \xrightarrow{p} 1$.

In the statement of the Theorem, $X \sim p_0$ independently of the data points used to estimate \widehat{T} . We will require a technical lemma that slightly generalizes Theorem 5.39 of ?.

Lemma 2. Assume A_1, \dots, A_m are independent zero-mean σ -subgaussian random vectors in \mathbb{R}^d with variances W_1, \dots, W_m respectively, and assume each $W_i \succeq \frac{1}{2} I_d$. Then with probability at least $1 - 2 \exp(-C_1 t^2)$, we have

$$\left\| \frac{1}{m} \sum_i A_i A_i' - W_i \right\|_{\text{op}} \leq \max(\delta, \delta^2), \quad \text{where } \delta = C_2 \sqrt{\frac{d}{m}} + \frac{t}{\sqrt{m}},$$

where the constants C_1 and C_2 depend on σ .

The proof of Lemma 2 is a straightforward modification of the argument in ?. We include it for self-containment, after the proof of Theorem 1.

Proof (Theorem 1). Because our method is invariant to invertible affine transformations of $S = (S_1, \dots, S_d)$, we can assume without loss of generality that $W = I$ and $\mu_0 = 0$. If the user selects basis \tilde{S} with mean $\tilde{\mu}_0 \neq 0$ and invertible variance $\tilde{W} \neq I_d$, we simply analyze the method using the affine transform $S = \tilde{W}^{-1/2}(\tilde{S} - \tilde{\mu}_0)$, which leads to the same estimator \hat{T} . If \tilde{W} is not invertible we can first discard redundant basis functions and then transform to a lower-dimensional S .

Note that if $\|\tilde{S}(x)\|_\infty$ (which is chosen by the analyst) is bounded then $\|S(x)\|_\infty$ is bounded as well, by some constant B_d that may grow with d ; therefore $|\eta' S(x)| < \sqrt{d} B_d$ and $S(X_{i,j}) - \mu_i$ is sub-gaussian with variance proxy less than $\sigma_d = dB_d^2$.

In that case

$$A = \frac{\tau^2}{m-1} cc' + I_d, \quad f(\beta) = \frac{\beta' A \beta}{\beta' \beta} = \frac{\tau^2}{m-1} \left(\frac{\beta' c}{\|\beta\|} \right)^2 + 1, \quad \text{and} \quad f(\beta^*) = \frac{\tau^2 \xi_d^2}{m-1} + 1, \quad (16)$$

where $\|c\| = \xi_d^2$, the optimal correlation using the d -basis.

Define the perturbations $Q = \hat{A} - A$ and $P = \hat{W} - I_d$. The result follows if we can prove that, for any fixed dimension d ,

$$\|Q\|_{\text{op}} = \|\hat{A} - A\|_{\text{op}} = o_p \left(\frac{\tau^2}{m-1} \right) \quad (17)$$

$$\|P\|_{\text{op}} = \|\hat{W} - I_d\|_{\text{op}} = o_p(1) \quad \left[= o_p \left(\frac{\tau^2}{m-1} \right) \right] \quad (18)$$

To see why, note that, for $\|P\|_{\text{op}} \leq 1/2$, we have

$$\sup_{\beta} \hat{f}(\beta) - f(\beta) = \sup_{\beta} \frac{\beta' A \beta + \beta' Q \beta}{\beta' \beta + \beta' P \beta} - \frac{\beta' A \beta}{\beta' \beta} \quad (19)$$

$$\leq \sup_{\beta} (1 + 2\|P\|_{\text{op}}) \frac{\beta' A \beta + \beta' Q \beta}{\beta' \beta} - \frac{\beta' A \beta}{\beta' \beta} \quad (20)$$

$$\leq 2\|P\|_{\text{op}} f(\beta^*) + (1 + 2\|P\|_{\text{op}}) \|Q\|_{\text{op}} \quad (21)$$

$$= o_p \left(\frac{\tau^2}{m-1} \right), \quad (22)$$

if (17) and (18) hold. There is a comparable bound for the other direction, leading to $\sup_{\beta} |\hat{f}(\beta) - f(\beta)| = o_p(\tau^2/(m-1))$. But because $f(\hat{\beta}) \geq f(\beta^*) - 2 \sup |\hat{f}(\beta) - f(\beta)|$, we have

$$\frac{\tau^2 \rho_{\hat{\beta}}^2}{m-1} + 1 \geq \frac{\tau^2 \rho_{\beta^*}^2}{m-1} + 1 + o_p \left(\frac{\tau^2}{m-1} \right) \quad \Rightarrow \quad \rho_{\hat{\beta}}^2 \geq \rho_{\beta^*}^2 + o_p(1).$$

Because $\rho_{\beta^*}^2 \rightarrow 1$ as $d \rightarrow \infty$, we have the desired consistency result as long as $d \rightarrow \infty$ slowly enough.

Throughout the proof we will let

$$\varepsilon_{i,j} = S(X_{i,j}) - \mu_i, \quad \bar{\varepsilon}_i = \bar{S}_i - \mu_i = \frac{1}{n_i} \sum_i \varepsilon_{i,j}, \quad \text{and} \quad \bar{\varepsilon} = \frac{1}{n_+} \sum_i n_i \bar{\varepsilon}_i = \frac{1}{n_+} \sum_{i,j} \varepsilon_{i,j}.$$

Recall the $\varepsilon_{i,j}$ are all mutually independent with sub-gaussian norms $\|\varepsilon_{i,j}\|_{\psi_2} \leq \sigma$; as a result $\|\bar{\varepsilon}_i\|_{\psi_2} \leq \sigma/n_i$ and $\|\bar{\varepsilon}\|_{\psi_2} = \sigma/n_+$.

Step 1. Bound $\|\widehat{A} - A\|_{\text{op}}$. We begin by writing $\bar{\mu} = \frac{1}{n_+} \sum_i n_i \mu_i = \mathbb{E}\bar{S}$. Then recalling that $\bar{S}_i - \bar{S} = (\mu_i - \bar{\mu}) + (\bar{\varepsilon}_i - \bar{\varepsilon})$, we can decompose

$$(\bar{S}_i - \bar{S})(\bar{S}_i - \bar{S})' = (\mu_i - \bar{\mu})(\mu_i - \bar{\mu})' + (\bar{\varepsilon}_i - \bar{\varepsilon})(\bar{\varepsilon}_i - \bar{\varepsilon})' \quad (23)$$

$$+ [(\bar{\varepsilon}_i - \bar{\varepsilon})(\mu_i - \bar{\mu})' + (\mu_i - \bar{\mu})(\bar{\varepsilon}_i - \bar{\varepsilon})']. \quad (24)$$

We can decompose the error in \widehat{A} as

$$Q = \widehat{A} - A = \left[\frac{1}{m-1} \sum_i n_i (\bar{S}_i - \bar{S})(\bar{S}_i - \bar{S})' \right] - \left[\frac{\sum_i n_i (\theta_i - \bar{\theta})^2}{m-1} cc' + I_d \right] \quad (25)$$

$$= \frac{1}{m-1} \sum_i n_i [(\mu_i - \bar{\mu})(\mu_i - \bar{\mu})' - (\theta_i - \bar{\theta})^2 cc'] \quad (26)$$

$$+ \left[\frac{1}{m-1} \sum_i n_i \text{Var}(\bar{S}_i - \bar{S}) \right] - I_d \quad (27)$$

$$+ \frac{1}{m-1} \sum_i n_i [(\bar{\varepsilon}_i - \bar{\varepsilon})(\bar{\varepsilon}_i - \bar{\varepsilon})' - \text{Var}(\bar{S}_i - \bar{S})] \quad (28)$$

$$+ \frac{1}{m-1} \sum_i n_i [(\bar{\varepsilon}_i - \bar{\varepsilon})(\mu_i - \bar{\mu})' + (\mu_i - \bar{\mu})(\bar{\varepsilon}_i - \bar{\varepsilon})'] \quad (29)$$

where $\bar{\mu} = \frac{1}{n_+} \sum_i n_i \mu_i$. Write $Q = Q_1 + Q_2 + Q_3 + Q_4$, for the four sums above. Then Q_1 is a deterministic error from the approximation $\mu_i \approx \mu_0 + \theta_i c = \theta_i c$, Q_2 is a deterministic error from the approximation $W_i \approx I_d$, and Q_3 and Q_4 are estimation errors.

Q_1 : Approximation Error for Means Recalling Lemma 3, write $\mu_i = \theta_i c + \theta_i \zeta(\theta_i)$, where the remainder $\zeta(\theta_i)$ satisfies

$$\zeta^*(\delta) = \sup\{\|\zeta(\theta)\| : |\theta| \leq \delta\} \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Furthermore because $|\bar{\theta}| \leq \max_j |\theta_j|$, we have $\|\mu_i - \bar{\mu} - (\theta_i - \bar{\theta})c\| \leq 2\zeta^*(\max_j |\theta_j|)$.

As a result we obtain

$$\|Q_1\|_{\text{op}} = \frac{1}{m-1} \sum_i n_i \|(\mu_i - \bar{\mu})(\mu_i - \bar{\mu})' - (\theta_i - \bar{\theta})^2 cc'\|_{\text{op}} \quad (30)$$

$$\leq \frac{1}{m-1} \sum_i n_i (2|\theta_i - \bar{\theta}| + \|\mu_i - \bar{\mu} - (\theta_i - \bar{\theta})c\|) \|\mu_i - \bar{\mu} - (\theta_i - \bar{\theta})c\| \quad (31)$$

$$\leq \frac{1}{m-1} \sum_i n_i \left(2|\theta_i - \bar{\theta}| + |\theta_i - \bar{\theta}| 2\zeta^*(\max_j |\theta_j|) \right) |\theta_i - \bar{\theta}| 2\zeta^*(\max_j |\theta_j|) \quad (32)$$

$$= \frac{\tau^2}{m-1} 4 \left[\zeta^*(\max_j |\theta_j|) + \zeta^*(\max_j |\theta_j|)^2 \right] \quad (33)$$

$$= o\left(\frac{\tau^2}{m-1}\right) \quad (34)$$

In (31) we use the fact that for $a, b \in \mathbb{R}^d$,

$$\|aa' - bb'\|_{\text{op}} = \frac{1}{2} \|(a+b)(a-b)' + (a-b)(a+b)'\|_{\text{op}} \quad (35)$$

$$\leq (\|a\| + \|b\|) \|a - b\| \quad (36)$$

$$\leq (2\|b\| + \|a - b\|) \|a - b\|, \quad (37)$$

and that $\|c\| \leq 1$.

Q₂: Approximation Error for Variances To compute the variance of $\bar{S}_i - \bar{S}$, we note that

$$\sum_i n_i (\bar{\varepsilon}_i - \bar{\varepsilon})(\bar{\varepsilon}_i - \bar{\varepsilon})' = \sum_i n_i \bar{\varepsilon}_i \bar{\varepsilon}'_i - n_+ \bar{\varepsilon} \bar{\varepsilon}', \quad (38)$$

leading to

$$\sum_i n_i \text{Var}(\bar{S}_i - \bar{S}) = \sum_i n_i \text{Var}(\bar{S}_i) - n_+ \text{Var}(\bar{S}) \quad (39)$$

$$= \sum_i W_i - \frac{1}{n_+} \sum_i n_i W_i \quad (40)$$

$$= \sum_i \left(1 - \frac{n_i}{n_+}\right) W_i. \quad (41)$$

Because we can also write $I_d = \frac{1}{m-1} \sum_i \left(1 - \frac{n_i}{n_+}\right) I_d$, and noting that by Lemma 3 we have $W_i = I_d + \theta_i D + o(|\theta_i|)$:

$$Q_2 = \left[\frac{1}{m-1} \sum_i \left(1 - \frac{n_i}{n_+}\right) W_i \right] - I_d \quad (42)$$

$$= \frac{1}{m-1} \sum_i \left(1 - \frac{n_i}{n_+}\right) (W_i - I_d) + I_d/m \quad (43)$$

Hence,

$$\|Q_2\|_{\text{op}} \leq \left| \frac{1}{m-1} \sum_i \left(1 - \frac{n_i}{n_+}\right) (\theta_i D + o(|\theta_i|)) \right|_{\text{op}} + \|I_d/m\|_{\text{op}} \quad (44)$$

$$= O(\max_i |\theta_i|) + 1/m \quad (45)$$

$$= o(1) = o\left(\frac{\tau^2}{m-1}\right), \quad (46)$$

Q₃: Estimation Error for Variances Applying (38) again, we can decompose Q_3 into two terms.

$$Q_3 = \frac{1}{m-1} \sum_i n_i [(\bar{\varepsilon}_i - \bar{\varepsilon})(\bar{\varepsilon}_i - \bar{\varepsilon})' - \text{Var}(\bar{S}_i - \bar{S})] \quad (47)$$

$$= \frac{1}{m-1} \sum_i (n_i \bar{\varepsilon}_i \bar{\varepsilon}'_i - W_i) - \frac{1}{m-1} \left[n_+ \bar{\varepsilon} \bar{\varepsilon}' - \frac{1}{n_+} \sum_i n_i W_i \right]. \quad (48)$$

Because $n_+ \|\bar{\varepsilon}\|^2 = \mathcal{O}_p(1)$, the operator norm of the second term is $\mathcal{O}_p(1/m)$. By Lemma 2, we have with probability at least $1 - 2 \exp\{-C_1 d\}$,

$$\left\| \frac{1}{m} \sum_i n_i \bar{\varepsilon}_i \bar{\varepsilon}'_i - W_i \right\|_{\text{op}} \leq \max(\delta, \delta^2), \quad \text{with} \quad \delta = (C_2 + 1) \sqrt{\frac{d}{m}}$$

Q₄: Estimation Error, Cross-Term We decompose Q_4 as $Q_4 = H + H'$, where

$$H = \frac{1}{m-1} \sum_i n_i (\bar{\varepsilon}_i - \bar{\varepsilon})(\mu_i - \bar{\mu})' \quad (49)$$

$$= \left(\frac{1}{m-1} \sum_i n_i \theta_i \bar{\varepsilon}_i \right) (\mu_i - \bar{\mu})' - \frac{n_+}{m-1} \bar{\varepsilon} (\mu_i - \bar{\mu})'. \quad (50)$$

Since $\sup_i \|\mu_i - \bar{\mu}\|_2 = O(\max_j |\theta_j|)$, $\|\sum_i n_i \bar{\varepsilon}_i\|_2 = O_p(d)$, and $\|n_+ \bar{\varepsilon}\|_2 = O_p(1)$ by Chebyshev's,

$$\|H + H'\|_{\text{op}} = O_p\left(d \max_j \theta_j^2\right) + O_p\left(\max_j |\theta_j|\right) = o_p\left(\frac{\tau^2}{m-1}\right) \quad (51)$$

Step 2: Bound $\|\widehat{W} - I\|$. We next bound $\|P\|_{\text{op}}$. We first decompose \widehat{W} into two variance components

$$\widehat{W} = \frac{1}{n_+ - m} \sum_{i,j} (S(X_{i,j}) - \bar{S}_i)(S(X_{i,j}) - \bar{S}_i)' \quad (52)$$

$$= \frac{1}{n_+ - m} \sum_{i,j} (\varepsilon_{i,j} - \bar{\varepsilon}_i)(\varepsilon_{i,j} - \bar{\varepsilon}_i)' \quad (53)$$

$$= \frac{1}{n_+ - m} \sum_{i,j} \varepsilon_{i,j} \varepsilon'_{i,j} - \frac{1}{n_+ - m} \sum_i n_i \bar{\varepsilon}_i \bar{\varepsilon}'_i. \quad (54)$$

This leads to a similar decomposition for the expectation:

$$\mathbb{E}\widehat{W} = \bar{W} = \frac{1}{n_+ - m} \sum_{i,j} W_i - \frac{1}{n_+ - m} \sum_i W_i.$$

Matching the terms and applying Lemma 2, we obtain that with probability at least $1 - 4 \exp\{-C_1 d\}$,

$$\|\widehat{W} - \bar{W}\|_{\text{op}} \leq \frac{n_+}{n_+ - m} \max(\delta_1, \delta_1^2) + \frac{m}{n_+ - m} \max(\delta_2, \delta_2^2),$$

with

$$\delta_1 = (C_2 + 1) \sqrt{\frac{d}{n_+}}, \quad \text{and} \quad \delta_2 = (C_2 + 1) \sqrt{\frac{d}{m}}.$$

Furthermore,

$$\bar{W} - I = \frac{1}{n_+ - m} \sum_i (n_i - 1)(W_i - I) \quad (55)$$

$$= \frac{1}{n_+ - m} \sum_i (n_i - 1) \theta_i D - o(|\theta_i|) \quad (56)$$

$$= \mathcal{O}(\max |\theta_i|) \rightarrow 0 \quad (57)$$

□

C Proof of Lemma 2

We now turn to the proof of Lemma 2.

Proof (Lemma 2). Write $Q = \frac{1}{m} \sum_i n_i a_i a_i' - W_i$ for the matrix whose norm we are bounding and $u = \max(\delta, \delta^2)$. Also, let $\|\cdot\|_{\psi_1}$ and $\|\cdot\|_{\psi_2}$ denote the sub-exponential and sub-gaussian norms respectively.

We repeat here the covering argument in ?, but we must track the variance adjustment a bit more carefully in the concentration step. The proof proceeds in three parts. First, we discretize the unit sphere S^{d-1} with a $\frac{1}{4}$ -net N , which has cardinality $|N| \leq 9^d$. Second, we show concentration for each fixed vector $\beta \in S^{d-1}$. Third, we apply a union bound.

Step 1. Approximation. Using Lemma 5.4 of ?, we can write

$$\|Q\|_{\text{op}} \leq 2 \max_{\beta \in N} |\beta' Q \beta| = 2 \max_{\beta \in N} \left| \frac{1}{m} \sum_i (a'_i \beta)^2 - \beta' W_i \beta \right|$$

Step 2. Concentration. Now fix any $\beta \in S^{d-1}$, and let $Z_i = a'_i \beta$. The Z_i are independent sub-gaussian random variables with $\mathbb{E} Z_i^2 = \beta' W_i \beta$ and $\|Z_i\|_{\psi_2} \leq \sigma$; hence, $V_i = Z_i^2 - \beta' W_i \beta$ are independent centered sub-exponential random variables with $\|V_i\|_{\psi_1} \leq 4\sigma^2$ and

$$\sigma \geq \|Z_i\|_{\psi_2} \geq \sqrt{\frac{\beta' W_i \beta}{2}} \geq \frac{1}{2}.$$

Thus, applying a concentration inequality for sums of sub-exponential random variables, we obtain

$$\mathbb{P} \left[\left| \frac{1}{m} \sum_i V_i \right| \geq \frac{u}{2} \right] \leq 2 \exp \left[-\frac{c_1}{\sigma^4} (C^2 d + t^2) \right],$$

for some constant c_1 .

Step 3. Union Bound. Finally, we union bound over $\beta \in N$ to obtain

$$\mathbb{P} \left[\max_{\beta \in N} \|\beta' Q \beta\| \geq \frac{u}{2} \right] \leq 9^d \cdot 2 \exp \left[-\frac{c_1}{\sigma^4} (C^2 d + t^2) \right] \leq 2 \exp \left[-c_1 t^2 / \sigma^4 \right],$$

for $C = C_\sigma = \sigma^2 \sqrt{\log(9)/c_1}$. □

D Proof of Theorem 2

Proof (Theorem 2). To reduce notational clutter we re-name P_{m+1} and P_{m+2} as P_1 and P_2 , with P_i coming from density p_{θ_i} , $\theta_i = t_i / \sqrt{n_i}$. Let n be an index for the sequence of problems, with sample sizes $n_1(n), n_2(n) \rightarrow \infty$. Let P_0 denote the distribution represented by density p_0 . Finally, assume without loss of generality that the Fisher information is 1 at $\theta = 0$ (otherwise we could re-parameterize the family by scaling θ).

We will study the behavior of $\Delta_{\hat{T}}$ for a fixed pair t_1, t_2 by relating it to the null case where $t_1 = t_2 = 0$. Note that contiguity is satisfied in this problem: $P_1^{n_1}$ is contiguous to $P_0^{n_1}$ and $P_2^{n_2}$ is contiguous to $P_0^{n_2}$ because both are product measures under local alternatives at $\theta = 0$; hence $P_1^{n_1} \times P_2^{n_2}$ is contiguous to $P_0^{n_1+n_2}$.

Since the family is DQM at $\theta = 0$ in the interior of Ω , it follows that $\Delta_T \rightarrow \mathcal{N}(0, 1)$ under the null. Furthermore, the test that rejects when $\Delta_T > \Phi^{-1}(1 - \alpha)$, where Φ is the c.d.f. of a standard normal, is an asymptotically efficient test at level α by Theorem 15.14 and the obvious two-sample generalization to 15.5 in Van der Vaart (2000).

The DQM condition implies local asymptotic normality for the sequence of experiments. Since $\text{corr}_0(\hat{T}_n(X), T(X) | \hat{T}_n) \xrightarrow{P} 1$, it follows that $\Delta_{\hat{T}_n} - \Delta_T \rightarrow 0$ in probability under the null as $n \rightarrow \infty$, and therefore also under the alternative by contiguity. As a result the test that rejects for $\Delta_{\hat{T}_n} > \Phi^{-1}(1 - \alpha)$ has the same asymptotic efficiency as Δ_T . □